A note on q-rings

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A note on q-rings

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§ 1. Introduction

Let R be a commutative ring with 1, and let e be an idempotent element of R. An ideal I of R is called quasi-invertible for e (for short, q-invertible) iff

(i) $e \in II^{-1}$ and
(ii) $\mathrm{ann}\,(e) \subseteq \mathrm{ann}\,(I)$.

Equivalently,

(i') $II^{-1} = (e)$ and
(ii') $\mathrm{ann}\,(e) = \mathrm{ann}\,(I)$

where

$$I^{-1} = \{x \in R_S = K \mid xI \subseteq R\}$$
R_S = the total quotient ring of R
$$\mathrm{ann}\,(I) = \{x \in R \mid xI = (0)\}, [8].$$

Notice that if $e = 1$, then I is invertible. It is easy to see that I is q-invertible iff. I is invertible in a direct summand of R.

It is well known that a finitely generated ideal I of R is invertible iff I is projective and contains a regular element [7].

On the other hand, it is proved in [8] that a f.g. ideal I of R is q-invertible for e iff I is projective and contains an element a such that

$$\mathrm{ann}\,(a) = \mathrm{ann}\,(e) = \mathrm{ann}\,(I).$$

Now recall that a ring R is called a P.P. ring iff every principal ideal of R is projective [2]. Equivalently, R is a P.P. ring iff for each $a \in R$, there exists a unique idempotent $a' \in R$ such that

$$aa' = a, \text{ and}$$
$$\mathrm{ann}\,(a) = \mathrm{ann}\,(a')$$

see [1]. The author calls in [1] an element a having this property semi-regular with associated idempotent a' (1n the terminology of Griffin in [5], a is called a'-regular). It was shown in [5] that if a is a'-regular, then $\exists\, x \in R_S$ such that $ax = a'$.

It follows from this that if R is a P.P. ring, then every principal ideal of R is q-invertible for some idempotent of R.

In this note we study rings in which every ideal is q-invertible. We call a ring with this property a q-ring.

§ 2. Statements of results

Theorem 2.1: If R is a commutative P.P. ring, then every finitely generated projective ideal of R is q-invertible.

Theorem 2.2: Let R be a commutative ring, then R is a q-ring iff R is Noetherian and semi-hereditary.

Theorem 2.3: Let R be a commutative ring, then R is q-ring iff R is a finite direct sum of Dedekind domains.

Since every Dedekind domain satisfies property (n) for each positive integer n [3], we get the following:

Corollary 2.4: Let R be a commutative Noetherian semi-hereditary ring. Then R satisfies property (n) $\forall n$, i.e. for each positive integer n, and for all $a,b \in R$,

$$(a,b)^n = (a^n, b^n).$$

Note: We will show in a forthcoming paper that the above relation is satisfied by every projective ideal (a,b) in any commutative ring.

The following corollary follows from theorems 2.3, and 2.2.

Corollary 2.5: If R is Noetherian semi-hereditary ring, then every ideal of R is a finite product of prime ideals.

Theorem 2.6: Let R be a commutative P.P. ring which is integrally closed. Assume that for each $a,b \in R$, and for some positive integer $n \geqq 2$,

$$(a,b)^n = (a^n, b^n)$$

(property (n)), then R is semi-hereditary ring.

§ 3. Proofs

Proof of Theorem 2.1: Let I be a non-zero f. g. ideal of R generated by $(a_1, a_2, \dots a_n)$, and let

$$A = \text{ann}(I).$$

Since I is projective, then A is generated by an idempotent e. Therefore, to show that I is q-invertible, it is enough to show that e belongs to II^{-1}. If this is false, then let M be an ideal of R which has the following properties:

(1) $M \supseteq II^{-1}$
(2) $e \notin M$.

(3) M is maximal among all ideals having properties (1), (2). Such an ideal exists by Zorn's lemma. It is easy to check that M is maximal ideal and $1-e \in M$.

Now, I_M is projective (in fact invertible) in the local ring R_M, and hence I_M is principal ideal generated by $\frac{a}{t}$, say, $a \in I$, $t \notin M$ [9]. Therefore, for each i, $1 \leqq i \leqq n$, $\exists\, s_i \notin M$ such that

$$a_i\, s_i \in aR.$$

Let $s = s_1 s_2 \dots s_n$. Clearly $S \notin M$.

Since R is a P.P. ring, there exists an idemptotent $e_1 \in R$ such that

$$ae_1 = a \quad \text{and}$$
$$\text{ann}\,(a) = \text{ann}\,(e_1) \quad [1].$$

Now, it is easy to check that $a \neq 0$, and $e_1 \notin M$, $1-e_1 \in M$. Using the result of Griffin quoted in the introduction, there exists $x \in R_S$ such that $ax = e_1$. Therefore $xsa_i \in R$, and hence $xs \in I^{-1}$, moreover,

$$xsa = se_1 \in II^{-1} \subseteq M.$$

But this is a contradiction because $s \notin M$ and $e_1 \notin M$, and this completes the proof of theorem 2.1.

Proof of Theorem 2.2:

If R is a q-ring, then every ideal I of R is q-invertible, and hence is finitely generated and projective [8], hence R is Noetherian and semi-hereditary. The converse follows from theorem 2.1.

Proof of theorem 2.3:

If R is a finite direct sum of Dedekind domains, then it is easy to see that R is a q-ring. To prove the converse, we first show that for each maximal ideal M of R, there is no ideal between M and M^2. In fact if ann $(M) = (0)$, then M is invertible and it is easy to check that there is no ideal between M and M^2. On the other hand, if ann $(M) = (e) \neq (0)$, where $e^2 = e$, (remember that M is projective), then by [9], there exists a principal ideal L generated by $(1-e)$ such that $M \subseteq L$, and $ML = M$. But since M is maximal and $L \neq R$, then $L = M$ and $M^2 = M$. It follows now from [7, p. 207] that R is a finite direct sum of Dedekind domains and special primary rings. It is easy to see that each special primary ring in this case is a principal ideal domain and hence is Dedekind domain. This completes the proof.

Proof of theorem 2.6:

Let M be any maximal ideal of R. Since R is a P.P. ring, then R_M is an integral domain [2]. Moreover, since R is integrally closed and satisfies property (n), then it is easy to check that R_M is integrally closed and satisfies property (n). It follows from [4] that each non zero finitely generated ideal of R_M is invertible and hence is principal. Using an argument similar to the one given in the proof of theorem 2.1, we can show that every f. g. ideal of R is projective, hence R is semi-hereditary (we leave the details of the argument to the reader). This completes the proof.

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